Energy Fluxes in RMHD Turbulence

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Plasmas

Plasma is everywhere!

- Interstellar medium, Stars, Lab plasmas.

Important applications:

- Solar/Stellar activity, Space weather, Fusion energy.
- We want to model it as accurately as possible.
- Ideally use kinetic theory... but too many particles to follow...
- $\Rightarrow$ Fluid approximation: Compressible MHD model
- Widely applicable, but... very demanding...
- need more assumptions...
- but more assumptions means less applicable...
- Research Question: How can we identify which dynamics are captured and which are not?

More assumptions...

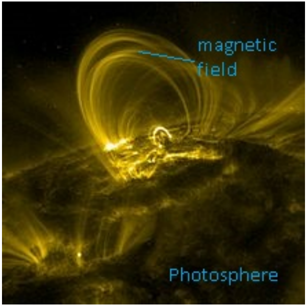


Image of coronal loops taken from https://scied.ucar.edu/sun-coronal-loops from TRACE/NASA with added text.

In many applications there is/are:

- Dominant magnetic field direction,
- Small $\|$ components,
- Small $\|$ gradients.

$\Rightarrow$ reduced MHD (RMHD) model...

- Origins in fusion
- What processes are captured?
- Can it be used in other fields?
- Coronal loops?
- Turbulence?

What is RMHD? (Kadomtsev74,Strauss76,Zank93)[3, 4, 5]

A non-linear, low frequency approximation of 3D MHD.

$$\underline{B} = \frac{B_0}{\varepsilon_{KP}}\underline{\hat{e}}_z + \underline{b}^{(1)} + \varepsilon_{KP}\underline{b}^{(2)} \ldots, \quad \underline{v} = \underline{v}^{(1)} + \varepsilon_{KP}\underline{v}^{(2)} \ldots,$$

Assumptions:

1. B_0 'large ',
   $\Rightarrow \ \varepsilon_{KP} = b^2/B_0^2 \ll 1.$
2. spectral anisotropy
   $\Rightarrow \ \varepsilon_S = k_{\|} \ll k_{\perp},$
3. variance anisotropy
   $\beta \lesssim 1$:
   $\Rightarrow \ \underline{B}_0 \cdot \underline{v} = \underline{B}_0 \cdot \underline{b} = 0,$

- $\nabla_{\perp} \cdot \underline{b} = 0, \quad \nabla_{\perp} \cdot \underline{v} = 0.$
- incompressible $\perp$ only (∂_z small).
- similar to (but not quite!) 2D MHD.

If $\beta \lesssim 1$: $b_z^{(1)} = v_z^{(1)} = 0,$

If $\beta \gg 1$: $b_z^{(1)}, v_z^{(1)} \neq 0$, passive scalars

The RMHD equations (leading order):

$$\partial_t \underline{v} + \underline{v} \cdot \nabla_{\perp} \underline{v} = \underline{b} \cdot \nabla_{\perp} \underline{b} + B_0 \partial_z \underline{b} - \nabla_{\perp} P + \nu \nabla_{\perp}^2 \underline{v},$$
$$\partial_t \underline{b} + \underline{v} \cdot \nabla_{\perp} \underline{b} = \underline{b} \cdot \nabla_{\perp} \underline{v} + B_0 \partial_z \underline{v} + \eta \nabla_{\perp}^2 \underline{b}.$$

Dynamics captured in RMHD?

- Problem: When is RMHD applicable?
- Aim: Identify processes included/neglected in RMHD.
- Method: Apply recent framework to RMHD and compare.
- HD framework: Johnson21 [2], MHD: Capocci25 [1].
- Splits energy flux into physically meaningful measurable terms:
  - Vortex stretching.
  - Strain rate self-amplification.
  - Current filament stretching.
  - Current sheet thinning.
- Process:
  - Gaussian filter: Remove small scales below $\ell \Rightarrow$ Focus on inertial scales.
  - Introduce subgridscale (SGS) stress tensor: $\tau^{\ell}(f_i, g_j) = \overline{f_i g_j}^{\ell} - \overline{f_i}^{\ell}\overline{g_j}^{\ell}$.
  - $\tau \Rightarrow$ function of field gradients.
  - Field gradients $\Rightarrow$ S Ω J Σ.

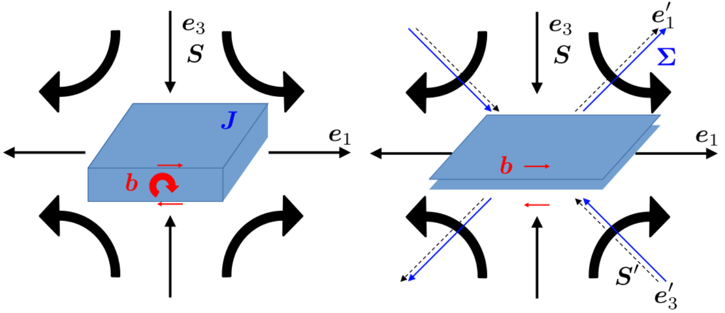
Incompressible MHD

$$\partial_t u_i + \partial_j(u_i u_j) = -\partial_i P + \partial_j(b_i b_j),$$
$$\partial_t b_i + \partial_j(b_i u_j) = \partial_j(u_i b_j),$$
$$\partial_i u_i = 0, \quad \partial_i b_i = 0.$$
$$\partial_t E_v + \nabla \cdot \mathcal{J}_v = -\Pi^{I,\ell} - \Pi^{M,\ell} - W^\ell,$$
$$\partial_t E_b + \nabla \cdot \mathcal{J}_b = -\Pi^{A,\ell} - \Pi^{D,\ell} + W^\ell$$

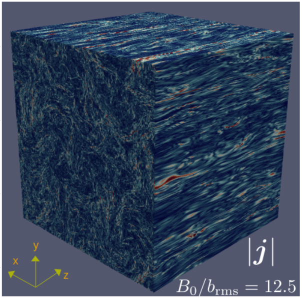
- Inertial: $\Pi^{I,\ell} = -\partial_j \overline{v_i}^\ell \tau^\ell(v_i, v_j)$.
- Maxwell: $\Pi^{M,\ell} = \partial_j \overline{v_i}^\ell \tau^\ell(b_i, b_j)$.
- Advection: $\Pi^{A,\ell} = -\partial_j \overline{b_i}^\ell \tau^\ell(b_i, v_j)$.
- Dynamo: $\Pi^{D,\ell} = \partial_j \overline{b_i}^\ell \tau^\ell(v_i, b_j)$.
- Energy Conversion: $W^\ell = \overline{b}_i^\ell \overline{b}_j^\ell \partial_j \overline{v}_i^\ell$.
- $> 0 \Rightarrow$ forward cascade.
- Π are further split into subfluxes

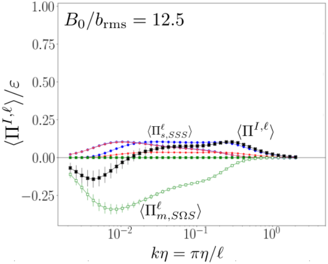
MHD Dynamics (Capocci2025)

- Subfluxes are triple product of combinations of $S,\ \Omega,\ J,\ \Sigma$.
- Example: Current sheet thinning - $\Pi^{M,\ \ell}_{s,\ SJ\Sigma} \propto Tr\left(\overline{S}^{\ell}\left(\overline{J}^{\ell}\overline{\Sigma}^{\ell} - \overline{\Sigma}^{\ell}\overline{J}^{\ell}\right)\right)$.



MHD Results (Capocci2025): Inertial Flux Π^I





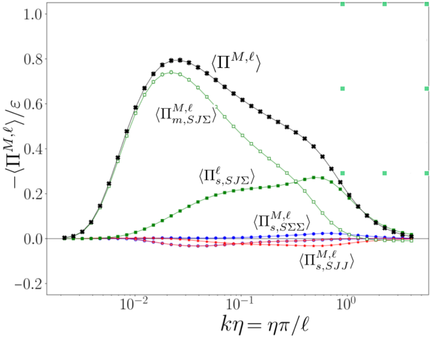
vortex thinning

$$\Pi^\ell_{m,S\Omega S} = -\int_0^{\ell^2} d\theta \; \mathrm{tr}\left(\overline{S}^\ell\left(\overline{\overline{S}^{\sqrt{\theta}}\,\overline{\Omega}^{\sqrt{\theta}}}^\phi - \overline{\overline{\Omega}^{\sqrt{\theta}}\,\overline{S}^{\sqrt{\theta}}}^\phi\right)\right)$$

$\Rightarrow$ Main contribution: Vortex thinning.

MHD Results (Capocci2025): Π^M

$$\Pi^{M,\ell} = \underbrace{\Pi^{M,\ell}_{s,S\Sigma\Sigma} + \Pi^{M,\ell}_{m,S\Sigma\Sigma}}_{\text{extensional/restoring effect of magnetic field}} + \underbrace{\Pi^{M,\ell}_{s,SJJ} + \Pi^{M,\ell}_{m,SJJ}}_{\text{strain amplification by current-filament stretching}} + \underbrace{\Pi^{M,\ell}_{s,SJ\Sigma} + \Pi^{M,\ell}_{m,SJ\Sigma}}_{\text{small-scale strain amplification by current-sheet thinning}}$$



$$\overline{\Sigma}^\ell = \left(\nabla \overline{\boldsymbol{b}}^\ell + (\nabla \overline{\boldsymbol{b}}^\ell)^t\right)/2$$

$$\overline{J}^\ell = \left(\nabla \overline{\boldsymbol{b}}^\ell - (\nabla \overline{\boldsymbol{b}}^\ell)^t\right)/2$$

Energy transfer almost exclusively

$$\Pi^{M,\ell}_{m,SJ\Sigma} = -\int_0^{\ell^2} d\theta \; \mathrm{tr}\left(\overline{S}^\ell \left(\overline{\overline{J}^{\sqrt{\theta}}\,\overline{\Sigma}^{\sqrt{\theta}}}^{\phi} - \overline{\overline{\Sigma}^{\sqrt{\theta}}}^{\phi}\,\overline{\overline{J}^{\sqrt{\theta}}}^{\phi}\right)\right)$$

$$\Pi^{M,\ell}_{s,SJ\Sigma} = -\ell^2 \mathrm{tr}\left(\overline{S}^\ell (\overline{J}^\ell \overline{\Sigma}^\ell - \overline{\Sigma}^\ell \overline{J}^\ell)\right)$$

⇒ Main contribution: Current sheet thinning and flow response.

Current work: Apply this to RMHD

The leading order RMHD equations:

$$\begin{aligned}
\partial_t v_i + \partial_j(v_i v_j) &= -\partial_{i,\,\neq z}(P) + \partial_j(b_i b_j) + B_0 \partial_z b_i, \\
\partial_t b_i + \partial_j(b_i v_j) &= \partial_j(v_i b_j) + B_0 \partial_z v_i. \\
\partial_j \cdot v_i &= \partial_j b_i = 0, \quad j = x, y. \\
\Rightarrow \partial_t E_v + \nabla \cdot \mathcal{J}_v &= -\Pi^{I,\ell} - \Pi^{M,\ell} - W^{\ell}_{\mathrm{RMHD}} - B_0 \partial_z \overline{v}^{\ell}_i \overline{b}^{\ell}_i, \\
\partial E_b + \nabla \cdot \mathcal{J}_b &= -\Pi^{A,\ell} - \Pi^{D,\ell} + W^{\ell}_{\mathrm{RMHD}} + B_0 \partial_z \overline{v}^{\ell}_i \overline{b}^{\ell}_i
\end{aligned}$$

- Same definitions, except $j \neq z$.
- W^{ℓ}_{RMHD} only contains $\perp$ gradients.
- $B_0 \partial_z$ is the only z derivative: dz small, B_0 large.
- In theory expect same dominant processes in RMHD.
- Vortex thinning more important as B_0 increases.

RMHD Dynamics

- Compare strain rate tensor for MHD and RMHD:

$$S_{\mathrm{MHD}} = \begin{bmatrix} \partial_x v_x & \frac{1}{2}\left(\partial_y v_x + \partial_x v_y\right) & \frac{1}{2}\left(\partial_z v_x + \partial_x v_z\right) \\ \frac{1}{2}\left(\partial_x v_y + \partial_y v_x\right) & \partial_y v_y & \frac{1}{2}\left(\partial_z v_y + \partial_y v_z\right) \\ \frac{1}{2}\left(\partial_x v_z + \partial_z v_x\right) & \frac{1}{2}\left(\partial_y v_z + \partial_z v_y\right) & \partial_z v_z \end{bmatrix},$$

$$S_{\mathrm{RMHD}} = \begin{bmatrix} \partial_x v_x & \frac{1}{2}\left(\partial_y v_x + \partial_x v_y\right) & \frac{1}{2}\partial_x v_z \\ \frac{1}{2}\left(\partial_x v_y + \partial_y v_x\right) & \partial_y v_y & \frac{1}{2}\partial_y v_z \\ \frac{1}{2}\partial_x v_z & \frac{1}{2}\partial_y v_z & 0 \end{bmatrix}.$$

- Low-β $\sim$ 2D
  - Dominated by vortex and current sheet thinning
- Terms involving $\parallel$ components $\Rightarrow$ high-β.

Summary

Identify dynamics that are captured in RMHD.

- Low β:
  - $\sim$ 2D as in Capocci2025.
  - Nonzero subfluxes:
    - Vortex thinning: $\Pi^{I,\ \ell}_{s,\ S\Omega S}$
    - Current sheet thinning and flow response: $\Pi^{M,\ \ell}_{SJ\Sigma}$
  - Also dominant in full 3D MHD using DNS.
    - Vortex thinning becomes stronger as B_0 increases.
- Current work $\Rightarrow$ High β:
  - Expect current filament stretching to become more important.
  - Understand how the $\parallel$ components affect the dynamics.

Thank you!

Questions? Comments? Feedback?

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